Cambridge International AS & A Level

Mathematics 9709

Paper 1 Pure Mathematics 1

Topic 8-Integration

Question No (32)

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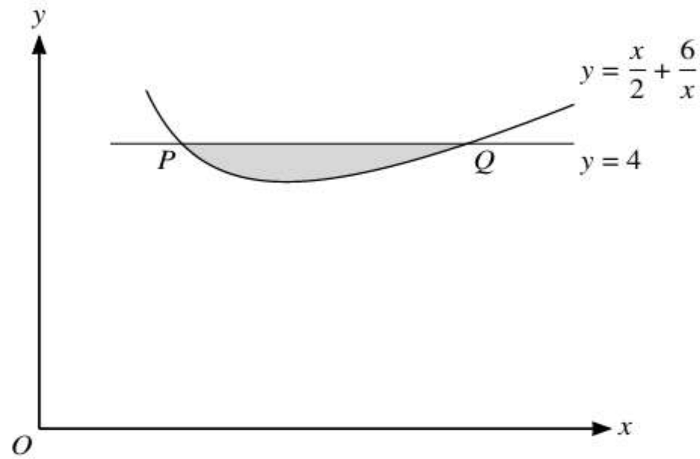

The diagram shows part of the curve $y = \frac{x}{2} + \frac{6}{x}$. The line $y = 4$ intersects the curve at the points P and Q.

(i) Show that the tangents to the curve at P and Q meet at a point on the line $y = x$.

(ii) Find, showing all necessary working, the volume obtained when the shaded region is rotated through 360° about the x-axis. Give your answer in terms of π.

<u>Solution</u>

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Equation of curve

$$y=\frac{x}{2}+\frac{6}{x} \longrightarrow (1)$$

As line $y=4$ intersect Equation (1), so

$$4=\frac{x}{2}+\frac{6}{x}$$

$$4=\frac{x^2+12}{2x}$$

$$8x=x^2+12$$

$$x^2-8x+12=0$$

By factorization

$$x^2-6x-2x+12=0$$

$$x(x-6)-2(x-6)=0$$

$$(x-6)(x-2)=0$$

$$x-6=0, \quad x-2=0$$

$$x=6, \quad x=2$$

$\therefore$ coordinates of $P(2,4)$ and coordinates of $Q(6,4)$.

From equation (1)

$$y=\frac{x}{2}+\frac{6}{x}$$

$$y=\frac{x}{2}+6x^{-1}$$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{1}{2} + 6\left((-1)\, x^{-1-1}\right)$$

$$= \frac{1}{2} - \frac{6}{x^2}$$

Gradient of tangent at $P(2,4)$

$$\frac{dy}{dx} = \frac{1}{2} - \frac{6}{(2)^2}$$

$$= \frac{1}{2} - \frac{6}{4}$$

$$= \frac{1}{2} - \frac{3}{2}$$

$$\frac{dy}{dx} = -1$$

Equation of tangent at $P(2,4)$

$$y - y_1 = \frac{dy}{dx}(x - x_1)$$

$$y - 4 = -1\,(x-2)$$

$$y - 4 = -x + 2$$

$$y = -x + 2 + 4$$

$$y = -x + 6 \longrightarrow (2)$$

Gradient of tangent at $Q(6,4)$

$$\frac{dy}{dx} = \frac{1}{2} - \frac{6}{x^2}$$

$$\frac{dy}{dx} = \frac{1}{2} - \frac{6}{(6)^2}$$

$$= \frac{1}{2} - \frac{1}{6}$$

$$\frac{dy}{dx} = \frac{3-1}{6} = \frac{2}{6} = \frac{1}{3}$$

Equation of tangent at $Q(6,4)$

$$y - y_1 = \frac{dy}{dx}(x - x_1)$$

$$y - 4 = \frac{1}{3}(x - 6)$$

$$y = \frac{x}{3} - 2 + 4$$

$$y = \frac{x}{3} + 2 \longrightarrow (3)$$

solving Equation (2) and (3)

$$-x + 6 = \frac{1}{3}x + 2$$

$$6 - 2 = \frac{1}{3}x + x$$

$$4 = \frac{4}{3}x$$

$$x = 3$$

put $x = 3$ in (2)

$$y = -3 + 6$$

$$y = 3$$

$\therefore$ Two tangents at P and Q meet at point $(3,3)$ which lies on the line $y = x$.

(ii) let $y_1 = 4$ and $y_2 = \frac{x}{2} + \frac{6}{x}$

volume of shaded region $= \pi \int_2^6 \left((y_1)^2 - (y_2)^2\right) dx$

(subtracting y_2 = equation of curve is at bottom in the given diagram)

$$= \pi \int_2^6 \left[(4)^2 - \left(\frac{x}{2} + \frac{6}{x} \right)^2 \right] dx$$

$$= \pi \int_2^6 \left[16 - \left(\frac{x^2}{4} + 2\left(\frac{x}{2}\right)(6/x) + \frac{36}{x^2} \right) \right] dx$$

$$= \pi \int_2^6 \left(16 - \frac{x^2}{4} - 6 - 36 x^{-2} \right) dx$$

$$= \pi \left| 16x - \frac{1}{4} \frac{x^{2+1}}{2+1} - 6x - 36 \frac{x^{-2+1}}{-2+1} \right|_2^6$$

$$= \pi \left| 16x - \frac{1}{12} x^3 - 6x + \frac{36}{x} \right|_2^6$$

$$= \pi \left[\left(16(6) - \frac{1}{12}(6)^3 - 6(6) + \frac{36}{6} \right) - \left(16(2) - \frac{1}{12}(2)^3 - 6(2) + \frac{36}{2} \right) \right]$$

$$= \pi \left[96 - 18 - 36 + 6 - 32 + \frac{8}{12} + 12 - 18 \right]$$

$$= \pi \left[96 - 18 - 36 + 6 - 32 + \frac{2}{3} + 12 - 18 \right]$$

$$= \pi \left[10 + \frac{2}{3} \right]$$

$$= \pi \quad \frac{32}{3}$$

$$V = 10 \frac{2}{3} \pi \text{ unit}^3$$